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Algebraic aspects of linear differential and difference equations

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Chapter 3

Shidlovskii irreducibility

3.1 Introduction

In this paper the notions Siegel normality and Shidlovskii irreducibility will be discussed. Being Siegel normal and being Shidlovskii irreducible are interesting properties of systems of ordinary linear differential equations, which arise in transcendental number theory. The aim of this article is to show how these properties can be characterized in terms of D-modules and the standard representation of the differential Galois group and how these characterizations can be used to verify Siegel normality or Shidlovskii irreducibility in some concrete practical examples.

The notion Siegel normality has been studied by F. Beukers, W.D. Brownawell and G. Heckman. Their important paper [BBH88] was the main source of inspiration for the research concerning the notion Shidlovskii irreducibility which led to this paper. Some interesting remarks on the notion Shidlovskii irreducibility are made in Bertrands paper [Ber90].

The following theorem is a fundamental theorem in the branch of transcendental number theory, which has been developed by C.L. Siegel and A.B. Shidlovskii. The theorem was proved by the latter one in 1959 (see [Shi89], Chapter 3).

Theorem 3.1.1 *Consider the $n \times n$ system of linear differential equations*

$$(A): \quad \frac{d}{dz}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

where $a_{ij} \in \mathbf{C}(z)$ for all i, j. Assume that:

1. *$(f_1(z), \ldots, f_n(z))^t$ is a solution of the system (A).*
2. *The component functions $f_1(z), \ldots, f_n(z)$ are all E-functions.*
3. *The component functions $f_1(z), \ldots, f_n(z)$ are homogeneous algebraic independent over $\mathbf{C}(z)$.*

4. ξ is a nonzero algebraic number and ξ is not a pole of one of the a_{ij}'s.

Then the numbers $f_1(\xi), \ldots, f_n(\xi)$ are homogeneous algebraic independent over $\mathbf{Q}$.

There exists also a more quantitative version of this theorem, namely if one adds the assumption that system (A) is homogeneous algebraic Siegel normal or that system (A) is homogeneous algebraic Shidlovskii irreducible, then one may conclude that the numbers $f_1(\xi), \ldots, f_n(\xi)$ are homogeneous algebraic independent with an effective measure of homogeneous algebraic independence. So this quantitative version of Shidlovskii's fundamental theorem motivates the study of Siegel's normality criterion and Shidlovskii's weaker irreducibility criterion.

The paper is organized in the following manner. In section 3.2 some notation will be fixed and some basic facts from D-modules and differential Galois theory will be recalled. Section 3.3 contains a summary of the results of Beukers, Brownawell and Heckman. In section 3.4 the results concerning the notion Shidlovskii irreducibility will be discussed. Some nice examples will be given in section 3.5.

Finally I wish to thank M. van der Put for his advice and interest. And I also would like to express my gratitude to F. Beukers, who called my attention to the book [Shi89] and gave me some ideas.

3.2 Preliminaries

Definition 3.2.1 *A differential field (K, δ) is a field K equipped with a derivation $\delta : K \to K$. That is $\delta(a+b) = \delta a + \delta b$ and $\delta(ab) = \delta(a)b + a\delta(b)$. The field of constants of K is $C = C_K = \{c \in K \mid \delta c = 0\}$.*

From now on we will assume that *char* $K =$ *char* $C = 0$ and that C is algebraically closed. For the purposes mentioned in the introduction one should take $K = \mathbf{C}(z)$, $C = \mathbf{C}$ and $\delta = \frac{d}{dz}$.

Definition 3.2.2 *Let $M \supseteq K$ and $L \supseteq K$ be differential fields. A field isomorphism $\phi : L \to M$ is a differential K-isomorphism, if $\phi(a) = a$ for all $a \in K$ and $\phi(\delta a) = \delta\phi(a)$ for all $a \in L$. If $M = L$ then ϕ is a differential K-automorphism.*

Consider the system of linear differential equations $(A) : \delta\mathbf{y} = A\mathbf{y}$, where A is a $n \times n$-matrix with coefficients in K.

Definition 3.2.3 *Differential field (L, δ^L) is called a Picard-Vessiot extension of K associated with (A) if*

1. *$L \supseteq K$ and $\delta^L|_K = \delta$.*

2. *$C_L = C_K = C$.*

3. *(A) has n linear independent solutions over C in L^n.*

4. *L is minimal with respect to the conditions 1,2 and 3 or equivalently if $U = (u_{ij})_{i,j=1,\ldots,n} \in L^{n\times n}$ is a fundamental matrix of the system (A) then $L = K(u_{11}, \ldots, u_{nn})$.*

Theorem 3.2.4 *For every system of linear differential equations (A) there exists a Picard-Vessiot extension L and this extension is unique up to differential K-isomorphism.*

Definition 3.2.5 *The differential Galois group $DGal(L/K)$ is the group consisting of all the differential K-automorphisms of L.*

If $U \in L^{n\times n}$ is a fundamental matrix of system (A) and $\sigma \in DGal(L/K)$, then it's obvious that also $\sigma(U)$ is a fundamental matrix of system (A). Hence

$$\sigma(U) = \begin{pmatrix} \sigma(u_{11}) & \cdots & \sigma(u_{1n}) \\ \vdots & & \vdots \\ \sigma(u_{n1}) & \cdots & \sigma(u_{nn}) \end{pmatrix} = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix} \cdot T_\sigma ,$$

where $T_\sigma \in Gl(n, C)$. So the elements of the differential Galois group act as C-linear maps on the space of solutions $V = \{c_1\mathbf{u}_1 + \cdots + c_1\mathbf{u}_n \mid c_1, \ldots, c_n \in C\}$. ($\mathbf{u}_i = (u_{1i}, \ldots, u_{ni})^t \in L^n$). But even a stronger statement holds.

Theorem 3.2.6 *$DGal(L/K)$ is a linear algebraic group over the field of constants C and*

$$dim_C DGal(L/K) = deg\, tr_K L = deg\, tr_K(u_{11}, \ldots, u_{nn}).$$

(By *deg tr* we mean degree of transcendence.) Further the Galois correspondence is of importance.

Theorem 3.2.7 *Suppose $G = DGal(L/K)$. Then the following statements holds:*

1. $(\forall \sigma \in G : \sigma(a) = a) \Rightarrow a \in K$.

2. *If H is an algebraic subgroup of G such that $K = \{a \in L \mid \forall \sigma \in H : \sigma(a) = a\}$ then $H = G$.*

3. *There is a 1-1 correspondence between algebraic subgroups H and differential subfields M.*

$$H = DGal(L/M) \Leftrightarrow M = \{a \in L \mid \forall \sigma \in H : \sigma(a) = a\}.$$

4. *Under this correspondence normal subgroups $H \subseteq G$ correspond to Picard-Vessiot extensions $M \supseteq K$ and vice versa. And then we have*

$$DGal(M/K) = G/H.$$

The first proofs of the last two theorems and the existence and uniqueness up to differential K-isomorphism of the Picard-Vessiot extension were given by E.R. Kolchin. (See [Kol 73], and [Lev90].) For more information about differential Galois theory we refer to [Kap57] and [Sin89].

Let $D = K\langle\partial\rangle$ be a skew polynomial ring consisting of all expressions $\sum_{i=0}^{k} a_i\partial^i$ with $a_i \in K$ for $i = 1, \ldots, n$. The multiplication in D is completely fixed by the relation $\partial a = a\partial + \delta a$ if $a \in K$. To a system of linear differential equations $(A) : \delta\mathbf{y} = A\mathbf{y}$ with $A \in K^{n\times n}$ we associate a left D-module $M = K^n$ (In the sequel we mean by D-module M a left D-module M with $dim_K M < \infty$.) in the following manner. If $m \in M$ then we define

$$(\sum_{i=0}^{k} a_i\partial^i) \cdot m := \sum_{i=0}^{k} a_i(\delta - A)^i \cdot m.$$

Conversely it's possible to associate a system of linear differential equations to a D-module M with a fixed K-base $E = \{e_1, \ldots, e_n\}$ in a natural way. Namely if $E = \{e_1, \ldots, e_n\}$ is a K-base of D-module M then there exist $a_{ij} \in K$ such that $\partial e_i = -\sum_{j=1}^{n} a_{ij}e_j$ for $j = 1, \ldots, n$. Let $A = (a_{ij})_{i,j=1,\ldots,n}$. If $m = \sum_{i=1}^{n} m_i e_i \in M$ then $\partial(m) = \sum_{i=1}^{n} \delta(m_i)e_i - \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}m_i e_j$. Hence

$$\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \longmapsto \begin{pmatrix} \delta(m_1) \\ \vdots \\ \delta(m_n) \end{pmatrix} - A \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$$

describes ∂ in coordinates. Now the system (A) corresponding to the matrix A is the system of linear differential equations associated to D-module M with fixed K-base E. Let $(\tilde{A})$ be a system of linear differential equations associated to the same D-module with an other fixed K-base $\tilde{E} = \{\tilde{e}_1, \ldots, \tilde{e}_n\}$ and let $\tilde{A} \in K^{n\times n}$ be the corresponding matrix. Suppose $\tilde{e}_i = \sum_{j=1}^{n} t_{ij}e_j$ for $i = 1, \ldots, n$, where $t_{ij} \in K$ for all i, j. Then the matrix $T = (t_{ij})_{i,j=1,\ldots,n} \in K^{n\times n}$ is invertible and we have the following relation.

$$\tilde{A} = TAT^{-1} + \delta(T)T^{-1}.$$

The systems (A) and $(\tilde{A})$ corresponding to the matrices A and $\tilde{A}$ are defined to be equivalent if the above relation holds for a certain invertible matrix $T \in K^{n\times n}$. In that case if $U \in L^{n\times n}$ is a fundamental matrix of (A) then $TU \in L^{n\times n}$ is a fundamental matrix of the system $(\tilde{A})$. Now it's obvious that the solution spaces V, $\tilde{V}$ of the systems (A), $(\tilde{A})$ are equivalent as representation spaces of the differential Galois group $DGal(L/K)$. And it is also clear that two Picard-Vessiot extensions L, $\tilde{L}$ associated with two equivalent systems $(A), (\tilde{A})$ are differential

K-isomorphic. Hence it is allowed to write $DGal(M)$ instead of $DGal(L/K)$ if L is a Picard-Vessiot extension associated with D-module M.

Recapitulating, we can view systems of linear differential equations as D-modules with a fixed K-base and D-modules as an equivalence class of systems of linear differential equations. We have introduced D-modules, because we prefer to study some properties of systems of linear differential equations K-base independently.

Let $L = L_M$ be a Picard-Vessiot extension associated with the D-module M. Then $L \otimes_K M$ becomes a $L\langle\partial\rangle$-module if we define $\partial(a \otimes m) = \delta^L(a) \otimes m + a \otimes \partial(m)$. Then $V = V_M = ker(\partial, L \otimes_K M)$ is the vector space of solutions on which the differential Galois group $G = DGal(M) = DGal(L/K)$ faithfully acts. We note that $L \otimes_C V = L \otimes_K M$, because if the elements $v_1, \ldots, v_n \in V$ are linear independent over C, then it is not difficult to demonstrate that they are also linear independent over L. The differential Galois group acts on $L_M \otimes_C V$ by $\sigma(a \otimes v) = \sigma(a) \otimes \sigma(v)$ if $\sigma \in G$. By taking the G-invariant elements of $L_M \otimes_C V$ we recover M, i.e. $M = (L_M \otimes_C V)^G$. Moreover there is a 1-1 correspondence between D-submodules $\tilde{M} \subseteq M$ and G-stable subspaces $\tilde{V} \subseteq V$.

$$\tilde{M} \longmapsto L \otimes_K \tilde{M} \longmapsto ker(\partial, L \otimes_K \tilde{M}) = \tilde{V}_{\tilde{M}}$$

$$\tilde{V} \longmapsto L_{\tilde{M}} \otimes_C \tilde{V} \longmapsto (L_{\tilde{M}} \otimes_C \tilde{V})^G = \tilde{M}_{\tilde{V}}$$

Because of this correspondence it's possible to replace M, N, D-(sub)modules and K by V, W, G-stable (sub)spaces in the next definitions and lemma's of this section. We denote $M \simeq N$ if M and N are isomorphic as D-modules and $V \simeq W$ if V and W are equivalent as representation spaces of G.

Definition 3.2.8 *D-module M is simple if there is no D-submodule $\tilde{M}$ such that $\{0\} \subset \tilde{M} \subset M$.*

(In this paper we use the symbol $\subset$ exclusively to denote a strict inclusion.)

Definition 3.2.9 *D-module M is undecomposable if there exist no D-submodules M_1, M_2 such that $\{0\} \subset M_1, M_2 \subset M$ and $M = M_1 \oplus M_2$.*

The next two classical lemmas will be used in this paper. Sometimes even tacitly!

Lemma 3.2.10 *(Krull-Schmidt). Let M be a D-module, $M = \bigoplus_{i=1}^{k} M_i$ and $M = \bigoplus_{j=1}^{l} \tilde{M}_j$, where the M_i and the $\tilde{M}_j$ are nontrivial undecomposable D-submodules. Then $k = l$ and there is a permutation $\pi \in S_k$ such that $M_i \simeq \tilde{M}_{\pi(i)}$ for $i = 1, \ldots, k$.*

If N, M are D-modules and $N \subset M$, then there exist a sequence of D-modules $N = N_0 \subset N_1 \subset \cdots \subset N_k = M$ such that N_i/N_{i-1} is a simple D-module for $i = 1, \ldots, k$. Such a sequence is called a Jordan-Hölder sequence from N to M.

Lemma 3.2.11 *(Jordan-Hölder). Let M be a D-module and let $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_k = M$ and $\{0\} = \tilde{M}_0 \subset \tilde{M}_1 \subset \cdots \subset \tilde{M}_l = M$ be Jordan-Hölder sequences. Then $l = k$ and there is a permutation $\pi \in S_k$ such that $M_i/M_{i-1} \simeq \tilde{M}_{\pi(i)}/\tilde{M}_{\pi(i)-1}$ for $i = 1, \ldots, k$*

Definition 3.2.12 *Let $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_k = M$ be a Jordan-Hölder sequence from $\{0\}$ to M and let S be a simple D-module then we define $mult_S M := \#\{i \in \{1, \ldots, k\} \mid M_i/M_{i-1} \simeq S\}$.*

The correctness of this definition is an immediate consequence of the Jordan-Hölder lemma.

Suppose M is a D-module. Let M^* denote its dual space, that is the the vector space consisting of all the K-linear maps $l : M \to K$. It is possible to give M^* a D-module structure. Define $(\partial^* l)(m) = \delta(l(m)) - l(\partial(m))$ for all $m \in M$ if $l \in M^*$.

Definition 3.2.13 *D-modules M and $\tilde{M}$ are cogredient if there exists a one dimensional D-module N such that $M \simeq N \otimes_K \tilde{M}$ and they are contragredient if there exists a one dimensional D-module N such that $M \simeq N \otimes_K \tilde{M}^*$.*

3.3 Siegel normality

This section contains a brief description of the results concerning Siegel normality in [BBH88].

The definition of the notion of Siegel normality is rather subtle. Consider the square matrix $A := (a_{ij})_{i,j=1,\ldots,n} \in K^{n \times n}$. Sometimes the matrix $A = (a_{ij})_{i,j=1,\ldots,n}$ splits into r submatrices

$$A_t = (a_{ij})_{i,j=1,\ldots,n_t}, \qquad t = 1, \ldots, r, \qquad n_1 + \cdots + n_r = n.$$

That is the square submatrices A_t are located along the main diagonal of A and all of the entries outside these submatrices are zero:

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix}.$$

We denote by (A) the system of differential equations which corresponds to the matrix A.

Definition 3.3.1 *System (A) is called linear Siegel normal if for any solution* $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r)^t$ *of (A) ,* $\mathbf{f}_i = (f_{i1}, \ldots, f_{in_i})^t$ *and any* $\mathbf{p}_i \in K^{n_i}$ *the relation* $\mathbf{p}_1\mathbf{f}_1 + \cdots + \mathbf{p}_r\mathbf{f}_r = 0$ *implies for each* $i = 1, \ldots, r$ *that either* $\mathbf{p}_i = 0$ *or* $\mathbf{f}_i = 0$.

If $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r)^t$ is a solution of system (A) then we let $\mathbf{f}^{l_1,\ldots,l_r}$ denote a $N_{l_1,\ldots,l_r}$-tuple consisting of all the $N_{l_1,\ldots,l_r}$ monomials

$$f_{11}^{j_{11}} \cdots f_{1n_1}^{j_{1n_1}} f_{21}^{j_{21}} \cdots f_{2n_2}^{j_{2n_2}} \cdots f_{r1}^{j_{r1}} \cdots f_{rn_r}^{j_{rn_r}}$$

in the component functions of the solution $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r)^t$ with $\sum_{k=1}^{n_i} j_{ik} = l_i$ for $i = 1, \ldots, r$. (The order of components in this $N_{l_1,\ldots,l_r}$-tuple doesn't matter.)

Definition 3.3.2 *System (A) is called homogeneous algebraic Siegel normal if for any* $N \geq 1$ *and any solution* $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_r)^t$ *of (A) and any* $\mathbf{p}_{l_1,\ldots,l_r} \in K^{N_{l_1,\ldots,l_r}}$

$$\sum_{l_1+\cdots+l_r=N} \mathbf{p}_{l_1,\ldots,l_r}\mathbf{f}^{l_1,\ldots,l_r} = 0$$

implies for all r-tuples $l_1, \ldots, l_r$ *with* $l_1 + \ldots + l_r = N$ *that either* $\mathbf{p}_{l_1,\ldots,l_r} = 0$ *or* $\mathbf{f}^{l_1,\ldots,l_r} = 0$.

These definitions can be translated easily into terms of D-modules.

Definition 3.3.3 *Let* $E = \{e_1, \ldots, e_n\}$ *be a K-base of the D-module M. Say* $M = \bigoplus_{i=1}^{r} M_i$ *, where the* M_i *are D-submodules with* $M_i = span_K(E_i)$, $E_i \subset E$. *We assume that r is taken as large as possible. Now M is called linear Siegel normal with respect to this K-base E if for all* $v \in V = ker(\delta, L \otimes_K M)$, *say* $v = \sum_{i=1}^{r} v_i \in V$ *with* $v_i \in L \otimes_K M_i$ *for* $i = 1, \ldots, r$ *and for all K-linear maps* $l : M \to K$ *extended as L-linear map* $(L \otimes_K M) \to L$ *(i.e.* $l \in (L \otimes_K M^*)^G$*) we have* $l(v) = 0$ *implies for all i:* $v_i = 0$ *or* $l(E_i) = \{0\}$.

Definition 3.3.4 *Let* $E = \{e_1, \ldots, e_n\}$ *be a K-base of the D-module M. Then M is called homogeneous algebraic Siegel normal with respect to the K-base E if* S^tM *(t-th symmetric tensor power) is linear Siegel normal with respect to the K-base* $S^tE = \{e_1^{t_1} \otimes_s e_2^{t_2} \otimes_s \cdots \otimes_s e_n^{t_n} \mid t_i \in \mathbf{N}_{\geq 0}, \sum_{i=1}^{n} t_i = t\}$ *for each* $t \geq 1$.

Unfortunately being Siegel normal is a property, which is not invariant under base transformations, therefore the following base independent definition is added.

Definition 3.3.5 *D-module M is called linear (homogeneous algebraic resp.) Siegel normal if there exists a K-base E of M such that M is linear (homogeneous algebraic resp.) Siegel normal with respect to this K-base E.*

Theorem 3.3.6 *Let M be a D-module. Then the following statements are equivalent:*

1. *M is linear Siegel normal.*
2. *The D-submodules M_i are simple for $i = 1, \dots, r$ and $M_i \not\simeq M_j$ if $i \neq j$.*
3. *The G-stable subspaces $V_i = ker(\partial, L \otimes_K M_i)$ are simple for $i = 1, \dots, r$ and $V_i \not\simeq V_j$ if $i \neq j$.*

Proof. The equivalence *2* $\Leftrightarrow$ *3* is an immediate consequence of the 1-1 correspondence between D-submodules $\tilde{M} \subset M$ and G-stable subspaces $\tilde{V} \subset V$. A proof of *1* $\Leftrightarrow$ *2* will be given.

1 $\Rightarrow$ *2* . Suppose M_i is not simple, then there exists a D-submodule $\tilde{M_i}$ with $\{0\} \subset \tilde{M_i} \subset M_i$. Let l be a K-linear map $M \to K$ with $\tilde{M_i} = ker(l)$. Extend l as L-linear map $L \otimes_K M \to L$. Let $\tilde{V_i} = ker(\partial, L \otimes_K \tilde{M_i})$ and let $\tilde{v_i} \in \tilde{V_i} \setminus \{0\}$. Then one has $l(\tilde{v_i}) = 0$, $\tilde{v_i} \neq 0$ and $l(M_i) \neq 0$. Hence M is not linear Siegel normal if M_i is not simple.

Suppose $M_i \simeq M_j$. Then there exists a K-linear map $\phi : M_i \to M_j$ such that $\forall q \in D : \phi(q.m_i) = q.\phi(m_i)$. Extend ϕ as L-linear map $L \otimes_K M_i \to L \otimes_K M_j$. Let l_i be a K-linear map $M \to K$ extended as L-linear map $L \otimes_K M \to L$ such that $l_i(M_k) = \{0\}$ if $k \neq i$ and $l_i(v_i) \neq 0$ for a $v_i \in V_i$. Let $v_j = \phi(v_i)$. Define $l_j = l_i \circ \phi$ and $l = l_i - l_j$. Then $l(v_i + v_j) = 0$, $v_i \neq 0$, $v_j \neq 0$, $l(M_i) \neq \{0\}$ and $l(M_j) \neq \{0\}$. Hence M is not linear Siegel normal if $M_i \simeq M_j$ and $i \neq j$.

2 $\Rightarrow$ *1*. Let $l : M \to K$ be a K-linear map. Extend l as a L-linear map $(L \otimes_K M) = (L \otimes_C V) \to L$ and define $W = \{v \in V \mid l(v) = 0\}$. Clearly, W is a G-stable subspace of V. Consider also the corresponding D-submodule $N = (L \otimes_C W)^G$. $N = \bigoplus_{i \in I} M_i$ with $I \subseteq \{1, \dots, r\}$, because the D-modules M_i are simple for $i = 1, \dots, r$ and $M_i \not\simeq M_j$ if $i \neq j$. Of course $l(N) = \{0\}$ and also $l(M_i) = \{0\}$ for $i \in I$. Suppose $v \in V$, $v = \sum_{i=1}^{r} v_i$ with $v_i \in V_i = ker(\delta, L \otimes_K M_i)$ for $i = 1, \dots, r$. If $l(v) = 0$ then we have $v_i = 0$ for $i \in \{1, \dots, k\} \setminus I$ and $l(M_i) = 0$ for $i \in I$. Hence M is linear Siegel normal if the M_i are simple and mutual non-equivalent. □

The main results in [BBH88] are the next two theorems.

Theorem 3.3.7 *Let M be a D-module. Suppose $dim_K M = n \geq 2$. Further let $G = DGal(M)$. Then the following statements are equivalent:*

1. *M is simple and homogeneous algebraic Siegel normal.*
2. *G contains $Sl(n, C)$ or $Sp(n, C)$*

Theorem 3.3.8 *Let M be a D-module and let $G = DGal(M)$. Then the following statements are equivalent:*

1. *M is homogeneous algebraic Siegel normal.*

2. $M = (\bigoplus_{i=1}^{r} M_i) \bigoplus (\bigoplus_{j=1}^{s} N_j)$, *where the M_i are are non-cogredient and non-contragredient simple linear Siegel normal D-modules with $dim_K M_i \geq 2$ and the N_j are one dimensional D-modules satisfying the following condition:*

$$S^{k_1} N_1 \otimes \cdots \otimes S^{k_s} N_s \simeq N_{triv}$$

implies either $k_1, \ldots, k_s = 0$ or $\sum_{i=1}^{s} k_i \neq 0$. (Here N_{triv} denotes the trivial D-module, i.e. $N = Ke$ with $\partial e = 0$, and if $k \in \mathbf{Z}_{\leq -1}$ then $S^k N$ denotes the D-module $S^{-k} N^*$.)

3.4 Shidlovskii irreducibility

Consider the $n \times n$ system of linear differential equations

$$(A): \quad \frac{d}{dz}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

where $a_{ij} \in K$ for all i, j.

Definition 3.4.1 *System (A) is called linear Shidlovskii irreducible if for any solution $\mathbf{f} = (f_1, \ldots, f_n)^t$ of (A) and any $p_i \in K$ the relation $p_1 f_1 + \cdots + p_n f_n = 0$ implies for each $i = 1, \ldots, n$ that either $p_i = 0$ or $f_i = 0$. In other words system (A) is linear Shidlovskii irreducible if the nonzero components of every solution $\mathbf{f} = (f_1, \ldots, f_n)^t$ are linear independent over K.*

Definition 3.4.2 *System (A) is called homogeneous algebraic Shidlovskii irreducible if the nonzero components of any solution are homogeneous algebraic independent over K.*

Compare these definitions of Shidlovskii irreducibility with the corresponding definitions of Siegel normality. It is immediately clear that being linear (homogeneous algebraic resp.) Shidlovskii irreducible is a weaker property of systems of linear differential equations than being linear (homogeneous algebraic resp.) Siegel normal.

Translating above definitions into terms of D-modules we get:

Definition 3.4.3 *Let $E = \{e_1, \ldots, e_n\}$ be a K-base of the D-module M. M is called linear Shidlovskii irreducible with respect to the K-base E if for all K-linear maps $l : M \to K$ extended as L-linear map $L \otimes_K M \to L$ (i.e. $l \in (L \otimes_K M^*)^G$) and all $v = \sum_{i=1}^{n} v_i e_i \in V = ker(\partial, L \otimes_K M)$ we have $l(v) = 0 \Rightarrow \forall i : v_i = 0$ or $l(e_i) = 0$.*

Definition 3.4.4 *Let $E = \{e_1, \ldots, e_n\}$ be a K-base of the D-module M. Then M is called homogeneous algebraic Shidlovskii irreducible with respect to the K-base E if $S^t M$ is linear Shidlovskii irreducible with respect to the K-base $S^t E$ for each $t \geq 1$.*

And now the base independent definition:

Definition 3.4.5 *M is called linear (homogeneous algebraic resp.) Shidlovskii irreducible if there exists a K-base E such that M is linear (homogeneous algebraic resp.) Shidlovskii irreducible with respect to this K-base E.*

Theorem 3.4.6 *Let M be a D-module. Equivalent statements are:*

1. *M is linear Shidlovskii irreducible.*
2. *M has a K-base $E = \{e_1, \ldots, e_n\}$ such that for every D-submodule $N \subseteq M$ there exists a subset $E_N \subseteq E$ such that $N = span_K E_N$.*
3. *There are only finitely many D-submodules $N \subseteq M$.*
4. *V has a C-base $F = \{f_1, \ldots, f_n\}$ such that for for every G-stable subspace $W \subseteq V$ there exists a subset $F_W \subseteq F$ such that $W = span_C F_W$.*
5. *There are only finitely many G-stable subspaces $W \subseteq V$.*

Proof. The equivalence *3* $\Leftrightarrow$ *5* can be proved easily by the 1-1 correspondence between D-submodules $\tilde{M} \subseteq M$ and G-stable subspaces $\tilde{V} \subseteq V$. The proofs of *2* $\Leftrightarrow$ *3* and *4* $\Leftrightarrow$ *5* are completely analogous. A proof of *1* $\Leftrightarrow$ *2* $\Leftrightarrow$ *3* will be given.

1 $\Rightarrow$ *2*. Assume that D-module M is linear Shidlovskii irreducible with respect to the K-base $E = \{e_1, \ldots, e_n\}$. Suppose further that $v = \sum_{i=1}^{n} v_i e_i \in V$ and $E(v) = \{e_i \in E \mid v_i \neq 0\}$. We associate to v a D-submodule $M(v)$ which is the smallest D-submodule of M such that $L \otimes_K M(v)$ contains v. We note that

$$M(v) = (L \otimes_C span_C(G.v))^G$$

Consider a K-linear map $l : M \to K$ extended as L-linear map $L \otimes_K M = L \otimes_C V \to L$. Clearly, $l(v) = 0 \Rightarrow l(L \otimes_C span_C(G.v)) = \{0\}$. So we get:

$$l(M(v)) = \{0\} \Leftrightarrow l(v) = 0 \Leftrightarrow \forall_{e_i \in E(v)} l(e_i) = 0 \Leftrightarrow l(span_K E(v)) = 0.$$

The second equivalence holds because M is linear Shidlovskii irreducible with respect to the K-base E. Hence $M(v) = span_K E(v)$ for all $v \in V$. Further if $N \subseteq M$ is a D-submodule then $N = M(v_1) + \cdots + M(v_s)$ for certain $v_1, \ldots, v_s \in V$. From this we get that $E = \{e_1, \ldots, e_n\}$ is a K-base of M such that for each D-submodule N we have $N = span_K E_N$ for a subset $E_N \subseteq E$.

2 ⇒ 1. We assume that $E = \{e_1, \dots, e_n\}$ is a K-base of M which satisfies the property of statement *2*. For each $v \in V$ let $M(v) = (L \otimes_C span_C(G.v))^G$. By assumption there exists a subset $E(v) \subseteq E$ such that $M(v) = span_K E(v)$. If $v = \sum_{i=1}^{n} v_i e_i$ then $\forall i : e_i \in E \setminus E(v) \Rightarrow v_i = 0$ and if l is a K-linear map $l : M \to K$ extended as L-linear map $L \otimes_K M \to L$ then we have

$$l(v) = 0 \Rightarrow l(M(v)) = 0 \Rightarrow l(e_i) = 0 \text{ if } e_i \in E(v).$$

Hence M is linear Shidlovskii irreducible.

2 ⇒ 3. Trivial.

3 ⇒ 2. Define $\mathcal{N} = \{N \mid N \subseteq M \text{ is a } D-\text{submodule }\}$. Successively will be proved:

A. For any $N, \tilde{N}, Q \in \mathcal{N}$ with $N \supseteq Q$ and $\tilde{N} \supseteq Q$ we have: $N = \tilde{N}$ if and only if $mult_S(N/Q) = mult_S(\tilde{N}/Q)$ for all simple D-modules S.

B. $(\mathcal{N}, +, \cap)$ is a finite distributive lattice.

C. There exists a K-base $E = \{e_1, \dots e_n\}$ of M such that for all $N \in \mathcal{N}$ we have $N = span_K E_N$, where $E_N \subseteq E$.

A. $(\Rightarrow)$ Trivial. $(\Leftarrow)$ We will prove this by induction with respect to the length k of the Jordan-Hölder sequence from Q to N. If $k = 0$ there is nothing left to prove. Assume the statement holds for $k = l - 1$. Let $Q = N_0 \subset N_1 \subset \cdots \subset N_l = N$ and $Q = \tilde{N}_0 \subset \tilde{N}_1 \subset \cdots \subset \tilde{N}_l = \tilde{N}$ be two Jordan-Hölder sequences from Q to N respectively from Q to $\tilde{N}$. Let q be minimal with respect to the condition that $\tilde{N}_q/\tilde{N}_{q-1} \simeq N_1/N_0$. The existence of such a q is evident because $mult_{N_1/N_0}(\tilde{N}/Q) = mult_{N_1/N_0}(N/Q) \geq 1$. From the Jordan-Hölder lemma we get $\tilde{N}_{q-1} + N_1 \supset \tilde{N}_{q-1}$, because $mult_{N_1/N_0}(\tilde{N}_{q-1}/Q) = 0 < 1 = mult_{N_1/N_0}((\tilde{N}_{q-1} + N_1)/Q)$. So $\tilde{N}_{q-1} + N_1 = \tilde{N}_q$ because the finiteness of the number of submodules of M implies that any quotient of M cannot contain two copies of the same submodule. Hence $N \supseteq N_1, \tilde{N} \supseteq N_1$ and for all simple D-modules we have $mult_S(N/N_1) = mult_S(\tilde{N}/N_1)$. Now we can apply the induction hypothesis and conclude $N = \tilde{N}$.

B. We have to prove the distributivity of the lattice $\mathcal{N}$. First we will prove

$$mult_S(N + \tilde{N}) = max(mult_S N, mult_S \tilde{N})$$

if S is a simple D-module and $N, \tilde{N} \in \mathcal{N}$. We will prove $mult_S(N + \tilde{N}) = max(mult_S N, mult_S \tilde{N})$ by induction with respect to the the length h of the Jordan-Hölder sequences from $\{0\}$ to N. If $h = 0$ everything is clear. Assume the statement holds for $h = k - 1$. Let $\{0\} = N_0 \subset N_1 \subset \cdots \subset N_k = N$ be a Jordan-Hölder sequence from $\{0\}$ to N. As a consequence of the proof of statement A) we have $N_1 \subseteq \tilde{N}$ if $mult_{N_1}\tilde{N} \geq 1$. Hence

$$\begin{aligned}
& mult_S(N+\tilde{N}) \\
= \; & mult_S(N+(\tilde{N}+N_1)) \\
= \; & mult_S(N/N_1+(\tilde{N}+N_1)/N_1)+\delta_{SN_1} \\
= \; & max(mult_S N/N_1, mult_S(\tilde{N}+N_1)/N_1)+\delta_{SN_1} \\
= \; & max(mult_S N, mult_S(\tilde{N}+N_1)) \\
= \; & max(mult_{N_1}N, mult_{N_1}\tilde{N}),
\end{aligned}$$

where $\delta_{SN_1}=0$ if $S \not\simeq N_1$ and $\delta_{SN_1}=1$ if $S \simeq N_1$.

The proof of $mult_S(N\cap\tilde{N}) = min(mult_S N, mult_S\tilde{N})$ for each simple D-module S and any $N,\tilde{N} \in \mathcal{N}$ is dual analogous. (In that case Jordan-Hölder sequences from N to M and from $\tilde{N}$ to M have to be considered.)

Let $N,\tilde{N},\hat{N} \in \mathcal{N}$. Then we have for each simple D-module S

$$\begin{aligned}
& mult_S N\cap(\tilde{N}+\hat{N}) \\
= \; & min(mult_S N, mult_S(\tilde{N}+\hat{N})) \\
= \; & min(mult_S N, max(mult_S\tilde{N}, ord_S\hat{N})) \\
= \; & max(min(mult_S N, mult_S\tilde{N}), min(mult_S N, mult_S\hat{N})) \\
= \; & max(mult_S(N\cap\tilde{N}), mult_S(N\cap\hat{N})) \\
= \; & mult_S((N\cap\tilde{N})+(N\cap\hat{N}))
\end{aligned}$$

Now we get as a consequence from A) that $\forall N,\tilde{N},\hat{N} \in \mathcal{N} : N\cap(\tilde{N}+\hat{N}) = (N\cap\tilde{N})+(N\cap\hat{N})$. In the same way it is possible to demonstrate that $\forall N,\tilde{N},\hat{N} \in \mathcal{N} : N+(\tilde{N}\cap\hat{N}) = (N+\tilde{N})\cap(N+\hat{N})$. Hence $(\mathcal{N},+,\cap)$ is a finite distributive lattice.

C. Suppose $\mathcal{N} = \{M_0, M_1 \ldots, M_s\}$ where $M_i \subset M_j \Rightarrow i<j$. Thus in particular $M_0=\{0\}$ and $M_s=M$. For any $i \in \{0,\ldots,s\}$ a set E_i satisfying the next two conditions will be constructed.

i. E_i is a K-base of D-module $\sum\limits_{j=0}^{i} M_j$.

ii. $\forall j \in \{0,\ldots,i\} : M_j := span_K(E_{M_j})$ with $E_{M_j} \subseteq E_i$.

Define $E_0 := \emptyset$. Of course E_0 satisfies the above conditions. Suppose $i \geq 1$. Now we assume that for all $j \in \{0,\ldots,i-1\}$ a set E_j satisfying the conditions i) and ii) has been constructed. If $M_i = M_k+M_l$ with $0<k,l<i$, then define $E_i := E_{i-1}$. Obviously, E_i satisfies the conditions i) and ii) if E_{i-1} satisfies these conditions. If $M_i \neq M_l+M_k$ for all k,l with $0<k,l<i$, then there exists a unique $h \in \{0,\ldots,i\}$ such that $M_h \subset M_i$ and M_i/M_h is a simple D-module. Suppose that $E_{M_i} = E_{M_h} \cup \tilde{E}_i$ (disjunct union) is a K-base of M_i, where $E_{M_h} \subseteq E_{i-1}$ is a K-base of M_h. Define $E_i := E_{i-1} \cup \tilde{E}_i$. Of course E_i satisfies condition ii). Now

we assume that E_i doesn't satisfy condition i) and derive a contradiction. In any case we have $span_K E_i = \sum_{j=0}^{i} M_j$. If the elements of E_i satisfy a linear dependence relation over K then there must be a $m \in (span_K E_{i-1} \cap span_K \tilde{E}_i) \setminus \{0\}$. In other words $0 \neq m \in (\sum_{j=0}^{i-1} M_j) \cap M_i$. But $m \notin span_K E_{M_h} \cap span_K \tilde{E}_i$. So $M_h \subset (\sum_{j=0}^{i-1} M_j) \cap M_i$. If $\sum_{j=0}^{i-1}(M_j \cap M_i) = M_i$ then $\sum_{j=0}^{i-1} M_j \supset M_i$ and so there exists $k, l : 0 < k < l < i$ such that $M_i = M_k + M_l$ (It is possible to choose M_k and M_l in such a way that M_i/M_k and M_i/M_l are simple D-modules.), but this is contradictory to an earlier assumption. Summarizing: if condition i) is not fulfilled then we have

$$M_h \subset (\sum_{j=0}^{i-1} M_j) \cap M_i = \text{(distributivity!)} \ \sum_{j=0}^{i-1}(M_j \cap M_i) \subset M_i,$$

contradicting that M_i/M_h is a simple D-module. Now we have obtained the desired contradiction and thus E_i satisfies condition i). We conclude that it is possible to construct successively sets E_i satisfying conditions i) and ii). Hence the set $E = E_s$ constructed this way is a K-base of M, which has the required property that $\forall N \in \mathcal{N} : N = span_K E_N$ with $E_N \subseteq E$. This finishes the proof of the theorem. □

Theorem 3.4.7 *Let M be a D-module. Equivalent statements are:*

1. *M is homogeneous algebraic Shidlovskii irreducible.*
2. *M has a K-base $E = \{e_1, \ldots, e_n\}$ such that for all $t \geq 1$ and for each D-module $N \subseteq S^t M$ there exists a subset $E_N \subseteq S^t E$ such that $N = span_K E_N$.*
3. *The vector space of solutions V has a C-base $F = \{f_1, \ldots, f_n\}$ such that for all $t \geq 1$ and for each G-stable subspace $W \subseteq S^t V$ there exists a subset $F_W \subseteq S^t F$. such that $W = span_C F_W$.*

Theorem 3.4.8 *Let $E = \{e_1, \ldots, e_n\}$ be a K-base of D-module M. Let $l_1, \ldots, l_n$ be K-linear maps $M \to K$, extended as L-linear maps $L \otimes_K M \to L$, such that $l_i(e_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. Then we have:*

1. *The following statements are equivalent:*
   - (a) *M is linear Shidlovskii irreducible with respect to the K-base E.*
   - (b) *The vector space of solutions V has a C-base $F = \{f_1, \ldots, f_n\}$ such that:*
     - i. *for every G-stable subspace $W \subseteq V$ there exists a subset $F_W \subseteq F$ such that $W = span_C(F_W)$.*

ii. for every G-stable subspace $W \subseteq V$, we have $f_i \in W, f_j \notin W \Rightarrow l_j(f_i) = 0$.

2. *The following statements are equivalent:*

   (a) *M is homogeneous algebraic Shidlovskii irreducible with respect to the K-base E.*

   (b) *The vector space of solutions V has a C-base $F = \{f_1, \dots, f_n\}$ such that:*

      i. for all $t \geq 1$ for every G-stable subspace $W \subseteq S^tV$ there exists a subset $F_W \subseteq S^tF$ such that $W = span_C(F_W)$.

      ii. for every G-stable subspace $W \subseteq V$, we have $f_i \in W, f_j \notin W \Rightarrow l_j(f_i) = 0$.

Proof. We will prove only statement *1.* $(a) \Rightarrow (b)$. Because of the 1-1 correspondence between G-stable subspaces $W \subset V$ and D-modules $N \subset M$ we can choose and number $f_1, f_2, \dots, f_n$ in such a way that together with condition i) the following condition is satisfied: $\forall I \subset \{1, \dots, n\} : span_K\{e_i\}_{i \in I} \subset M$ is a D-module $\Leftrightarrow span_C\{f_i\}_{i \in I}$ is a G-stable subspace. Let $W \subset V$ be a G-stable subspace, then there exists a D-module N such that $W = ker(\partial, L \otimes_K N)$. Suppose $N = span_K\{e_i\}_{i \in I}$ with $I \subset \{1, \dots, n\}$. Then $W = span_C\{f_i\}_{i \in I}$. If $j \notin I$ then $l_j(N) = 0$ and thus also $l_j(L \otimes_K N) = 0$ and $l_j(W) = 0$. From this we get $l_j(f_i) = 0$ if $i \in I$ and $j \notin I$.

$(b) \Rightarrow (a)$. We assume that V has a C-base $F = \{f_1, \dots, f_n\}$, which satisfies the conditions i) and ii). Let $v \in V$ and suppose that $span_C(G.v) = span_C\{f_i\}_{i \in I_v}$. Let l be a K-linear map $M \to K$, extended as L-linear map $L \otimes_K M \to L$ such that $l(v) = 0$. From ii) we get $l_j(v) = 0$ if $j \notin I_v$. Consider $M(v) = (L \otimes_C span_C(G.v))^G$. $M(v)$ is $\#I_v$-dimensional and $l_j(M(v)) = 0$ if $j \notin I_v$. So $M(v) = span_K\{e_i\}_{i \in I_v}$ and $l(M(v)) = 0 \Rightarrow l(e_i) = 0$ for $i \in I_v$. We conclude that M is linear Shidlovskii irreducible with respect to the K-base $E = \{e_1, \dots, e_n\}$, because $l_j(v) = 0$ if $j \notin I_v$ and $l(e_i) = 0$ if $i \in I_v$. $\square$

It is possible to formulate a theorem analogous to theorem 3.4.8 for Siegel normality.

Theorem 3.4.9 *Let M be a D-module and let $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_r = M$ be a Jordan-Hölder sequence. Then we have:*

1. *$\bigoplus_{i=1}^{r} M_i/M_{i-1}$ is linear Siegel normal $\Rightarrow$ M is linear Shidlovskii irreducible.*

2. *$\bigoplus_{i=1}^{r} M_i/M_{i-1}$ is homogeneous algebraic Siegel normal $\Rightarrow$ M is homogeneous algebraic Shidlov-
skii irreducible.*

Proof. *1.* Suppose M is not linear Shidlovskii irreducible then it is not difficult to show that there exist a quotient of M which contains two copies of the same simple submodule. Hence the graded module $\bigoplus_{i=1}^{r} M_i/M_{i-1}$ must contain two copies of the same simple submodule. But then $\bigoplus_{i=1}^{r} M_i/M_{i-1}$ is not linear Siegel normal because of theorem 3.3.6.

2. Suppose $\bigoplus_{i=1}^{r} M_i/M_{i-1}$ is homogeneous algebraic Siegel normal. Let $t_i \in \mathbf{N}$ for $i = 1, \ldots, r$. Then according to lemma 2.1 in [BBH88] D-module $S_{t_1,\ldots,t_r} = S^{t_1}(M_1/M_0) \otimes \cdots \otimes S^{t_r}(M_r/M_{r-1})$ is simple and further $S_{t_1,\ldots,t_r} \not\cong S_{\tilde{t}_1,\ldots,\tilde{t}_r}$ if $\sum_{i=1}^{r} t_i = \sum_{i=1}^{r} \tilde{t}_i$ and $(t_1, \ldots, t_r) \neq (\tilde{t}_1, \ldots, \tilde{t}_r)$. Obviously if S is a simple D-module then $mult_S S^t(\bigoplus_{i=1}^{r} M_i/M_{i-1}) = mult_S S^t M$ for all $t \in \mathbf{Z}_{\geq 0}$. Hence $S^t M$ is homogeneous algebraic Shidlovskii irreducible because $S^t(\bigoplus_{i=1}^{r} M_i/M_{i-1}) = \bigoplus_{\sum t_i = t} S_{t_1,\ldots,t_r}$. $\square$

We wish to remark that the converse of this theorem does not hold. For instance consider the $\mathbf{C}(z)\langle\partial\rangle$-module $M = \mathbf{C}(z)e_1 + \mathbf{C}(z)e_2$, with $\partial e_1 = 0$ and $\partial e_2 = \frac{1}{z}e_1$ and its submodule $M_1 = \mathbf{C}(z)e_1$. It is not difficult to verify that M is linear and homogeneous algebraic Shidlovskii irreducible, but $M_1 \oplus M/M_1$ is neither linear nor homogeneous algebraic Siegel normal.

3.5 Examples

In this section the results of the previous sections will be applied to get some concrete examples of systems of linear differential equations which are homogeneous algebraic Shidlovskii irreducible but which are not homogeneous algebraic Siegel normal. We restrict ourselves to systems of linear differential equations on which Shidlovskii's fundamental theorem is applicable (See section 1).

Let $0 \leq p < q$, $\mu_1, \ldots, \mu_p \in \mathbf{C}$ and $\lambda_1, \ldots, \lambda_q \in \mathbf{C} \setminus \mathbf{Z}_{\leq 0}$. Then we define the generalized hypergeometric function

$$\phi_{\mu_1,\ldots,\mu_p;\lambda_1,\ldots,\lambda_q}(z) = \sum_{n=0}^{\infty} \frac{(\mu_1)_n \cdots (\mu_p)_n}{(\lambda_1)_n \cdots (\lambda_q)_n} \left(\frac{z}{q-p}\right)^{(q-p)n},$$

where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. The function $\phi_{\mu_1,\ldots,\mu_p;\lambda_1,\ldots,\lambda_q}$ satisfies the q-th order differential equation

$$(\dagger) \qquad \left(\prod_{i=1}^{q}(\theta + r(\lambda_i - 1)) - z^r \prod_{i=1}^{p}(\theta + r\mu_i)\right) y = r^q(\lambda_1 - 1)\cdots(\lambda_q - 1)\,,$$

where $r = q - p$ and $\theta = z\frac{d}{dz}$. (See [Shi89], chapter 5, §1.) If $\mu_1, \ldots, \mu_p \in \mathbf{Q}$ and $\lambda_1, \ldots, \lambda_q \in \mathbf{Q} \setminus \mathbf{Z}_{\leq 0}$ then $\phi_{\mu_1,\ldots,\mu_p;\lambda_1,\ldots,\lambda_q}$ is an E-function

First we consider the special case $p = 0$ and $q = 1$. Suppose $\lambda \in \mathbf{C} \setminus \mathbf{Z}_{\leq 0}$ then $\phi_\lambda(z) = \sum_{n=0}^{\infty} \frac{1}{(\lambda)_n} z^n$ satisfies the linear differential equation

$$y' = \frac{z+1-\lambda}{z} y + \frac{\lambda - 1}{z} \qquad (' = \frac{d}{dz}).$$

Theorem 3.5.1 *Let $\lambda \in \mathbf{Q} \setminus \mathbf{Z}_{\leq 0}$. Suppose ξ is a nonzero algebraic number. Then $\phi_\lambda(\xi)$ is a transcendental number with an effective measure of transcendence.*

Proof. Consider the system of linear differential equations

$$(\hat{A}) : \quad \frac{d}{dz} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{z+1-\lambda}{z} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Let $\psi_\lambda(z) = z^{1-\lambda} e^z$. Then $\hat{U} = \begin{pmatrix} \psi_\lambda & 0 \\ 0 & 1 \end{pmatrix}$ is a fundamental matrix of the system $(\hat{A})$. Let $L = \mathbf{C}(z)(\psi_\lambda)$. Then L is a Picard-Vessiot extension of $\mathbf{C}(z)$ associated with the system $(\hat{A})$. If $\sigma \in DGal(L/\mathbf{C}(z))$ then $\sigma(\hat{U}) = \hat{U} . \begin{pmatrix} \chi(\sigma) & 0 \\ 0 & 1 \end{pmatrix}$ where χ is an character of the differential Galois group. For all $m \geq 1$ we have $\chi^m \neq 1$ because $\psi_\lambda^m \notin \mathbf{C}(z)$. Hence the system $(\hat{A})$ is homogeneous algebraic Siegel normal as a consequence of theorem 3.3.8 and so the system of linear differential equations

$$(\tilde{A}) : \quad \frac{d}{dz} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{z+1-\lambda}{z} & \frac{\lambda-1}{z} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

is homogeneous algebraic Shidlovskii irreducible because of theorem 3.4.9 . Further $(\phi_\lambda, 1)^t$ is a solution of the system $(\tilde{A})$ consisting of nonzero E-functions. Hence the numbers $\phi_\lambda(\xi), 1$ are homogeneous algebraic independent with an effective measure of homogeneous algebraic independence and so $\phi_\lambda(\xi)$ is transcendental with an effective measure of transcendence. $\square$

Theorem 3.5.2 *Let $\mu_1, \ldots, \mu_p \in \mathbf{Q}$, $\lambda_1, \ldots, \lambda_p \in \mathbf{Q} \setminus \mathbf{Z}$, $0 \leq p < q$. Suppose that $q \geq 2$ and that at least one of the following conditions holds.*

1. *$\lambda_i - \mu_j \notin \mathbf{Z}$ for $1 \leq i \leq p$, $1 \leq j \leq q$ and the sums $\lambda_i + \lambda_j$ $(1 \leq i \leq j \leq q)$ are all distinct modulo $\mathbf{Z}$.*

2. *$p = 0$, $q = 2$ or q is odd and there is not a permutation $\pi \in S_q$ and a divisor $d > 1$ of q such that $\lambda_i = \lambda_{\pi(i)} + \frac{1}{d}$ (mod $\mathbf{Z}$) for $i = 1, \ldots, q$.*

Let ξ be a nonzero algebraic number. Then the numbers

$$\phi_{\mu_1,\ldots,\mu_p;\lambda_1,\ldots,\lambda_q}(\xi), \ldots, \phi^{(p-1)}_{\mu_1,\ldots,\mu_p;\lambda_1,\ldots,\lambda_q}(\xi)$$

are algebraic independent with an effective measure of algebraic independence.

Proof. Consider the homogeneous linear differential equation

$$\left(\prod_{i=1}^{q}(\theta + r(\lambda_i - 1)) - z^r \prod_{i=1}^{p}(\theta + r\mu_i)\right) y = 0$$

The $q \times q$ system corresponding to this differential equation is simple and homogeneous algebraic Siegel normal. For a proof of this statement we refer to proposition 4.4 and the proof of theorem 4.5 in [BBH88]. Hence the $(q+1) \times (q+1)$ system of differential equations corresponding to (†) is homogeneous algebraic Shidlovskii irreducible. And so the numbers

$$\phi_{\mu_1,\ldots,\mu_p;\lambda_1,\ldots,\lambda_q}(\xi), \ldots, \phi^{(p-1)}_{\mu_1,\ldots,\mu_p;\lambda_1,\ldots,\lambda_q}(\xi)$$

are algebraic independent with an effective measure of algebraic independence. □

Let $\lambda, \mu \in \mathbf{C} \setminus \mathbf{Z}_{\leq -1}$. Then we define the function

$$K_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(\lambda+1)_n(\mu+1)_n}(\frac{z}{2})^{2n},$$

which satisfies the non-homogeneous second order differential equation

$$y'' + \frac{2\lambda + 2\mu + 1}{z} y' + (1 + \frac{4\lambda\mu}{z^2})y = \frac{4\lambda\mu}{z^2}$$

It is useful to define also the function $K_{\lambda,\mu;\xi}(z) = K_{\lambda,\mu}(\xi z)$, if $\lambda, \mu \in \mathbf{C} \setminus \mathbf{Z}_{\leq -1}$ and $\xi \in \mathbf{C}$. If $\lambda, \mu \in \mathbf{Q} \setminus \mathbf{Z}_{\leq -1}$ and ξ is an algebraic number then $K_{\lambda,\mu;\xi}$ is an E-function. (See [Shi89], chapter 5, §1.) The function $K_{\lambda,\mu;\xi}$ satisfies the equation

$$y'' + \frac{2\lambda + 2\mu + 1}{z} y' + (\xi^2 + \frac{4\lambda\mu}{z^2})y = \frac{4\lambda\mu}{z^2}.$$

Lemma 3.5.3 *Consider the corresponding homogeneous differential equation*

$$(\dagger) \quad y'' + \frac{2\lambda + 2\mu + 1}{z} y' + (\xi^2 + \frac{4\lambda\mu}{z^2})y = 0.$$

Let G be the differential Galois group over $\mathbf{C}(z)$ associated with this differential equation. If $\lambda - \mu + \frac{1}{2} \notin \mathbf{Z}$ and $\xi \neq 0$ then G contains $Sl(2, \mathbf{C})$.

Proof. Let $\overline{\mathbf{C}(z)}$ be the algebraic closure of $\mathbf{C}(z)$. We will prove that $\overline{G} = DGal((\dagger), \overline{\mathbf{C}(z)}) \simeq Sl(2, \mathbf{C})$. Hence G contains $Sl(2, \mathbf{C})$.

After transforming the differential equation by the substitution $x = z^{\lambda+\mu}y$ we get a new equation

$$(\ddagger) \quad x'' + \frac{1}{z}x' + (\xi^2 - \frac{(\lambda-\mu)^2}{x^2})y = 0.$$

The differential equations (†) and (‡) are equivalent over $\overline{\mathbf{C}(z)}$. It is known that under the conditions of this lemma $DGal((\ddagger), \mathbf{C}(z)) \simeq Sl(2, \mathbf{C})$. (See [Kol68].) $Sl(2, \mathbf{C})$ is connected and $[DGal((\ddagger), \mathbf{C}(z)) : DGal((\ddagger), \overline{\mathbf{C}(z)})] < \infty$. Hence $DGal((\ddagger), \overline{\mathbf{C}(z)}) \simeq Sl(2, \mathbf{C})$ and thus $\overline{G} \simeq Sl(2, \mathbf{C})$. □

Let $A_{\lambda,\mu;\xi} = \begin{pmatrix} 0 & 1 \\ -\xi^2 - \frac{4\lambda\mu}{z^2} & \frac{-2\lambda-2\mu-1}{z} \end{pmatrix}$. Then the 2×2 system of linear differential equations $(A_{\lambda,\mu;\xi}) : \ y' = A_{\lambda,\mu;\xi} y$ is the system corresponding to the second order linear differential equation (†).

Lemma 3.5.4 *Consider the sytems of linear differential equations* $(A_{\lambda_1,\mu_1;\xi_1})$ *and* $(A_{\lambda_2,\mu_2;\xi_2})$. *Suppose that* $\lambda_i - \mu_i + \frac{1}{2} \notin \mathbf{Z}$, $i = 1, 2$ *and* $\xi_1, \xi_2 \neq 0$. *Suppose further that the systems* $(A_{\lambda_1,\mu_1;\xi_1})$ *and* $(A_{\lambda_2,\mu_2;\xi_2})$ *are cogrediënt or contragrediënt. Then* $\xi_1^2 = \xi_2^2$ *and either* $(\lambda_1 - \mu_1) + (\lambda_2 - \mu_2) \in \mathbf{Z}$ *or* $(\lambda_1 - \mu_1) - (\lambda_2 - \mu_2) \in \mathbf{Z}$.

Proof. Let $U_1 = \begin{pmatrix} f_1 & g_1 \\ f_1' & g_1' \end{pmatrix}$ and $U_2 = \begin{pmatrix} f_2 & g_2 \\ f_2' & g_2' \end{pmatrix}$ be fundamental matrices of the systems $(A_{\lambda_1,\mu_1;\xi_1})$ and $(A_{\lambda_2,\mu_2;\xi_2})$. Then

$$3 \leq deg\, tr_{\mathbf{C}(z)}(f_1, f_1', g_1, g_1', f_2, f_2', g_2, g_2') \leq 4,$$

because the systems $(A_{\lambda_1,\mu_1;\xi_1})$ and $(A_{\lambda_2,\mu_2;\xi_2})$ are cogrediënt or contragrediënt and

$$deg\, tr_{\mathbf{C}(z)}(f_1, f_1', g_1, g_1') = dim_{\mathbf{C}} DGal((A_{\lambda_1,\mu_1,\xi_1}), \mathbf{C}(z)) = 3.$$

Let $\tilde{f}_i = z^{\lambda_i+\mu_i} f_i$ and $\tilde{g}_i = z^{\lambda_i+\mu_i} g_i$, $i = 1, 2$ and let $V_i = \begin{pmatrix} \tilde{f}_i & \tilde{g}_i \\ \tilde{f}_i' & \tilde{g}_i' \end{pmatrix}$, $i = 1, 2$, then $V_i = M_i U_i$, where $M_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $\lambda_i + \mu_i = 0$ and else $M_i = \begin{pmatrix} z^{\lambda_i+\mu_i} & 0 \\ z^{\lambda_i+\mu_i-1} & z^{\lambda_i+\mu_i} \end{pmatrix}$. It is easy to verify that V_i is a fundamental matrix of the system of differential equations $(B_i) : \ y' = \begin{pmatrix} 0 & 1 \\ -\xi_i^2 - \frac{(\lambda_i-\mu_i)^2}{z^2} & -\frac{1}{z} \end{pmatrix} y$. According to [Kol68] $deg\, tr_{\mathbf{Q}(z)}(\tilde{f}_1, \tilde{f}_1', \tilde{g}_1, \tilde{g}_1', \tilde{f}_2, \tilde{f}_2', \tilde{g}_2, \tilde{g}_2') < 6$ implies $\xi_1^2 = \xi_2^2$ and either $(\lambda_1 - \mu_1) + (\lambda_2 - \mu_2) \in \mathbf{Z}$ or $(\lambda_1 - \mu_1) - (\lambda_2 - \mu_2) \in \mathbf{Z}$. □

Theorem 3.5.5 *Suppose that* $\lambda_i, \mu_i \in \mathbf{Q} \setminus \mathbf{Z}_{\leq -1}$, $i = 1, \ldots, n$, $n \geq 1$, *satisfy the conditions* $\lambda_i - \mu_i + \frac{1}{2} \notin \mathbf{Z}$, *and* $(\lambda_{i_1} - \mu_{i_1}) + (\lambda_{i_2} - \mu_{i_2}) \notin \mathbf{Z}$, $(\lambda_{i_1} - \mu_{i_1}) - (\lambda_{i_2} - \mu_{i_2}) \notin \mathbf{Z}$, $i_1, i_2 = 1, \ldots, n$, $i_1 \neq i_2$. *Let* $\xi_1, \ldots, \xi_m$, $m \geq 1$, *be nonzero algebraic numbers such that* $\xi_i^2 \neq \xi_j^2$ *if* $i \neq j$. *Then the* $2mn$ *numbers* $K_{\lambda_i,\mu_i}(\xi_j), K'_{\lambda_i,\mu_i}(\xi_j)$, $i = 1, \ldots, n$ $j = 1, \ldots, m$, *are algebraic independent with an effective measure of algebraic independence.*

Proof. Consider the $(2mn + 1) \times (2mn + 1)$ system of linear differential equations

$$(\hat{A}): \quad y' = \begin{pmatrix} A_{\lambda_1,\mu_1;\xi_1} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & \cdots & A_{\lambda_i,\mu_i;\xi_j} & \cdots & 0 & 0 \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & A_{\lambda_n,\mu_n;\xi_m} & 0 \\ 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix} y.$$

System $(\hat{A})$ is homogeneous algebraic Siegel normal because of lemma 3.5.3, lemma 3.5.4 and theorem 3.3.8. Let $B_{i,j} = \begin{pmatrix} 0 \\ 4\lambda_i\mu_i \end{pmatrix}$ Then the system

$$(A): \quad y' = \begin{pmatrix} A_{\lambda_1,\mu_1;\xi_1} & \cdots & 0 & \cdots & 0 & B_{1,1} \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & \cdots & A_{\lambda_i,\mu_i;\xi_j} & \cdots & 0 & B_{i,j} \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & A_{\lambda_n,\mu_n;\xi_m} & B_{n,m} \\ 0 & \cdots & 0 & \cdots & 0 & 0 \end{pmatrix} y$$

is homogeneous algebraic Shidlovskii irreducible because of theorem 3.4.9.

The $2mn+1$-tuple $(K_{\lambda_1,\mu_1;\xi_1}, \ldots, K_{\lambda_i,\mu_i;\xi_j}, \ldots, K_{\lambda_n,\mu_n;\xi_m}, 1)^t$ is a solution of the system (A) consisting of nonzero E-functions. Hence the numbers

$$K_{\lambda_1,\mu_1;\xi_1}(1), \ldots, K_{\lambda_i,\mu_i;\xi_j}(1), \ldots, K_{\lambda_n,\mu_n;\xi_m}(1), 1$$

are homogeneous algebraic independent with an effective measure of homogeneous algebraic independence and so the numbers

$$K_{\lambda_1,\mu_1}(\xi_1), \ldots, K_{\lambda_i,\mu_i}(\xi_j), \ldots, K_{\lambda_n,\mu_n}(\xi_m)$$

are algebraic independent with an effective measure of algebraic independence. $\square$